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ON RECONSTRUCTION OF A STABLE PART OF BAND-LIMITED FUNCTION BY INTERPOLATION

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In the paper the interpolation problem of a band-limited function is discussed. The definition of stable and unstable (with respect to interpolation) parts of band-limited function is given. It is proved that the space of band-limited functions $\mathcal{B}$ splits into the sum $\mathcal{B} = \mathcal{S} \oplus \mathcal{U}$, where $\mathcal{S}$ consists of the functions of stable reconstruction, $\mathcal{U}$ consists of the functions of unstable reconstruction, $\dim \mathcal{U} < \infty$. The algorithms of the reconstruction of a stable part of band-limited function and of the nearest stable function are proposed.

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1. Let $f(x)$, $x \in \mathbb{R}^n$ be a square-integrable function which can be represented as a Fourier integral

$$f(x) = \frac{1}{(2\pi)^n} \int_R e^{i\langle x,y\rangle} F(y)\, dy \tag{1}$$

over the bounded domain $R \subset \mathbb{R}^n$. Here $\langle x, y\rangle = x_1 y_1 + \ldots + x_n y_n$. It is known by Paley-Wiener theorem that $f(x)$ extends to entire function in $\mathbb{C}^n \supset \mathbb{R}^n$.

Assume that $f(x)$ is known outside an open region $\Lambda \subset \mathbb{R}^n$. Consider the *interpolation problem:* to reconstruct the values of $f(x)$, $x \in \Lambda$ by $f\Big|_{\mathbb{R}^n \setminus \Lambda}$.

Such a problem appears in incomplete data inverse problems for Radon and spherical mean transforms [1], [2], [3]. Analogous problem has been considered in connection with the communication theory, radio engineering, optics [4], [5].

Since (1) is an entire function, the interpolation problem has unique solution, which can be found by any of known methods. The case is that each interpolation algorithm has exponential coefficient of noise increase [1], [2]. It makes unstable every practical algorithm and casts doubt on reliability of any reconstruction based on interpolation.

Following [6] we will try to obtain some *stable* procedure, which construct function $\tilde{f}$ close to the original f.

2. Denote by $\mathcal{B}_R$ the space of functions, which can be represented in (1) with $F \in L^2(\mathbb{R}^n)$. By Plancherel theorem $\mathcal{B}_R \subset L^2(\mathbb{R}^n)$, equipped with the standard scalar product. For $f \in \mathcal{B}_R$ by $\|f\|_\Lambda$ we mean a part of its energy in the domain Λ:

$$\|f\|_\Lambda^2 = \int_\Lambda |f(x)|^2\, dx.$$

Adopt the following definition:

D e f i n i t i o n 1. $f \in \mathcal{B}_R$ is a function of stable (unstable) reconstruction, if

$$\|f\|_\Lambda \leqslant \|f\|_{\mathbb{R}^n\setminus\Lambda} \qquad \Big(\|f\|_\Lambda > \|f\|_{\mathbb{R}^n\setminus\Lambda}\Big). \tag{2}$$

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It turns out that the space of band-limited functions splits into the sum of two subspaces: one consists of functions of stable reconstruction, another one—of unstable.

Theorem 2. *Let $\Lambda \subset \mathbb{R}^n$ be a bounded region. The the following decomposition holds:*

$$\mathcal{B}_R = \mathcal{S}_\Lambda \oplus \mathcal{U}_\Lambda,$$

where any function from $\mathcal{S}_\Lambda$ is of stable reconstruction, any function from $\mathcal{U}_\Lambda$ is of stable reconstruction, $\dim \mathcal{U}_\Lambda < \infty$.

Proof. Define two projections in $L^2(\mathbb{R}^n)$ (cf. [7]). The first one is *time-limiting operator:*

$$D : f(x) \mapsto \begin{cases} f(x), & x \in \Lambda, \\ 0, & x \notin \Lambda. \end{cases}$$

The second one is *band-limiting operator:*

$$B : f(x) \mapsto \frac{1}{(2\pi)^n} \int_R e^{i\langle x,\xi\rangle} F(\xi)\, d\xi,$$

where $F(\xi)$ is the Fourier transform of $f(x)$.

By the Krasichkov theorem [8], there exists a system

$$\{\psi_i\}, \quad i = 0, 1, \ldots \tag{3}$$

of the real-valued eigenfunctions of operator BD, which possesses the following properties:
1. The correspondent eigenvalues $\{\lambda_i\}$ are real, positive and ordered as follows: $\lambda_0 \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_i \to 0$.
2. $\{\psi_i\}$ is complete system in $\mathcal{B}_R$.
3. $\{\psi_i\}$ is orthonormal in $L^2(\mathbb{R}^n)$.
4. $\langle D\psi_i, \psi_j\rangle = \begin{cases} 0, & i \neq j, \\ \lambda_i, & i = j. \end{cases}$

The sought subspaces are the following: $\mathcal{S}_\Lambda = l.h.\{\psi_i | \lambda_i \leqslant 1/2\}$, $\mathcal{U}_\Lambda = l.h.\{\psi_i | \lambda_i > 1/2\}$. $\mathcal{B}_R = \mathcal{S}_\Lambda \oplus \mathcal{U}_\Lambda$.

The property 1 implies that $\dim \mathcal{U} < \infty$.

Any function $f \in \mathcal{B}_R$ can be represented as Fourier series over the system $\{\psi_i\}$, $f = \sum_i \alpha_i \psi_i$, where $\alpha_i = \langle f, \psi_i\rangle$. Therefore

$$\|f\|^2 = \sum \alpha_i^2,$$

$$\|f\|^2_\Lambda = \langle Df, f\rangle = \sum \alpha_i^2 \langle D\psi_i, \psi_i\rangle = \sum \lambda_i \alpha_i^2.$$

In a similar way

$$\|f\|^2_{\mathbb{R}^n \setminus \Lambda} = \sum (1 - \lambda_i)\alpha_i^2.$$

One can see that for any $f \in \mathcal{U}_\Lambda$

$$\|f\|^2_\Lambda - \|f\|^2_{\mathbb{R}^n \setminus \Lambda} = \sum_{\lambda_i > 1/2} (2\lambda_i - 1)\alpha_i^2 > 0,$$

so f is of unstable reconstruction.

Analogously, any function from $\mathcal{S}_\Lambda$ is of stable reconstruction. The theorem is proved.

Remark 3. In the case of dimension one the functions (3) coincide with prolate wave spheroidal functions [7]. In higher dimensions a little is knows about relation of functions (3) with any known special functions (cf. [9]).

3. The system (3) allows us to propose two methods of stable recovering of a part of band-limited function.

Definition 4. For a function $f \in \mathcal{B}_R$ by $f_{\mathcal{S}}$, *a stable part of* f, we mean the orthogonal projection of f onto $\mathcal{S}_\Lambda$.

Theorem 5. *A stable part of* $f \in \mathcal{B}_R$ *can be recovered by* $f\Big|_{\mathbb{R}^n\setminus\Lambda}$ *with the following formula:*

$$f_{\mathcal{S}} = \sum_{\lambda_i \leqslant 1/2} \frac{a_i}{1-\lambda_i}\psi_i,$$

where

$$a_i = \int_{\mathbb{R}^n\setminus\Lambda} f\psi_i\,dx. \tag{4}$$

This theorem in immediate corollary of properties of functions (3).

Note that inequality (2) defines a cone in the space $\mathcal{B}_R$. Infinite dimensional Lagrange factors principle brings us th following theorem, which gives a formula for reconstruction of *the nearest* with respect to L^2-metric stable function $f_{\mathcal{N}}$ (which does not in general coincide with $f_{\mathcal{S}}$).

Theorem 6. *The nearest to* $f \in \mathcal{B}_R$ *function of stable reconstruction can be recovered by* $f\Big|_{\mathbb{R}^n\setminus\Lambda}$ *with the following formula:*

$$f_{\mathcal{N}} = \sum \frac{a_i\lambda_i}{\lambda_i + \tau(2\lambda_i - 1)}\psi_i,$$

where a_i *are defined in (4),* τ *satisfies the following relation:*

$$\sum \frac{(2\lambda_i - 1)\lambda_i^2 a_i^2}{(\lambda_i + \tau(2\lambda_i - 1))^2} = 0$$

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